

Bethe vectors of $\mathfrak{gl}(3)$ -invariant integrable models, their scalar products and form factors

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This short note corresponds to a talk given at *Lie Theory and Its Applications in Physics*, (Varna, Bulgaria, June 2013) and is based on joint works with S. Belliard, S. Pakuliak and N. Slavnov, see [arXiv:1206.4931](#), [arXiv:1207.0956](#), [arXiv:1210.0768](#), [arXiv:1211.3968](#) and [arXiv:1312.1488](#).

LAPTH-Conf-003/14

1 General background

We first expose the general algebraic framework that will be needed for our calculation. This part is not new at all, it just recasts well-known facts from QISM approach, see e.g. [1, 2, 3] and references therein. We also use it to fix our notations.

1.1 R -matrix

As usual in integrable systems, the basic tool is the so-called R -matrix $R(x, y) \in V \otimes V$, where $x, y \in \mathbb{C}$ are the spectral parameters and $V = \text{End}(\mathbb{C}^N)$ is a vector space. $R(x, y)$ obeys the Yang-Baxter equation, written in $V \otimes V \otimes V$:

$$R^{12}(x_1, x_2) R^{13}(x_1, x_3) R^{23}(x_2, x_3) = R^{23}(x_2, x_3) R^{13}(x_1, x_3) R^{12}(x_1, x_2).$$

Here and below, we will use the auxiliary space notation: the superscripts indicate in which copies of V spaces R acts non trivially. For instance, in $V \otimes V \otimes V$, we have:

$$R^{12}(x, y) = R(x, y) \otimes \mathbb{I} \quad \text{and} \quad R^{23}(x, y) = \mathbb{I} \otimes R(x, y),$$

while in $V^{\otimes N}$, we would have:

$$R^{k, k+1}(x, y) = \mathbb{I}^{\otimes(k-1)} \otimes R(x, y) \otimes \mathbb{I}^{N-k-1}.$$

1.2 Monodromy and transfer matrices

We define the monodromy matrix

$$T(x) = \sum_{i,j=1}^N e_{ij} \otimes T_{ij}(x) \in \text{End}(\mathbb{C}^N) \otimes \mathcal{A}[[x^{-1}]],$$

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where e_{ij} is the elementary $N \times N$ matrix with 1 at position (i, j) . $T(x)$ obeys the commutation relations (or FRT relations)

$$R^{12}(x, y) T^1(x) T^2(y) = T^2(y) T^1(x) R^{12}(x, y). \quad (1)$$

Through these exchange relations, the monodromy matrix generates an algebra \mathcal{A} , defined by the choice of the R -matrix. Typically, \mathcal{A} is the Yangian $Y(\mathfrak{gl}_N)$ or the quantum affine group $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. The monodromy matrix leads to an integrable model through the transfer matrix

$$t(x) = \text{tr}_0 T^0(x) = \sum_{j=1}^N T_{jj}(x) \in \mathcal{A}[[x^{-1}]].$$

Integrability can be seen in the relation $[t(x), t(y)] = 0$, that is valid at the algebraic level (i.e. in the \mathcal{A} algebra), due to the relations (1).

In the following, we will deal with the Yangian $Y(\mathfrak{gl}_3)$, based on the $SU(3)$ -invariant R -matrix

$$R(x, y) = \mathbf{I} + g(x, y) \mathbf{P} \in \text{End}(\mathbb{C}^3) \otimes \text{End}(\mathbb{C}^3) \quad \text{and} \quad g(x, y) = \frac{c}{x - y},$$

where \mathbf{I} is the identity matrix, \mathbf{P} is the permutation matrix between two spaces $\text{End}(\mathbb{C}^3)$, and c is a constant. Note however that many properties will be also valid for the trigonometric R -matrix associated to the quantum group $\mathcal{U}_q(\widehat{\mathfrak{gl}}_3)$, and also for $Y(\mathfrak{gl}_N)$ or $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ algebras, see below.

1.3 Choice of a physical model

The choice of a representation for the algebra \mathcal{A} leads to a physical model. For instance, taking for the monodromy and transfer matrices, the usual form

$$t(x) = \text{tr}_0 T^0(x) = \text{tr}_0 R^{01}(x, 0) R^{02}(x, 0) \cdots R^{0L}(x, 0) \in (\text{End}(\mathbb{C}^N))^{\otimes L},$$

we get an Hamiltonian acting on L copies of the fundamental representation of \mathcal{A} , $(\mathbb{C}^N)^{\otimes L}$: it is the generalized \mathfrak{gl}_N -XXX or \mathfrak{gl}_N -XXZ closed spin chain with L sites.

To summarize this algebraic part, we have a two step procedure for the determination of a physical model:

- The choice of an R -matrix, that fixes the algebra we are dealing with, that is to say the interaction in the bulk of the spin chain (leading to XXX, XXZ, ... models);
- The choice of the "spin content" of the chain, that is given by the choice of the representations of the algebra, in our context the form of the monodromy matrix.

Here, as already stated, we will deal with $\mathcal{A} = Y(\mathfrak{gl}_3)$. However, to be as general (and algebraic) as possible, we will not fix the representation we act on, and just assume that it is highest weight:

$$T_{jj}(w)|0\rangle = \lambda_j(w)|0\rangle, \quad j = 1, 2, 3 \quad T_{ij}(w)|0\rangle = 0, \quad 1 \leq i < j \leq 3$$

for some arbitrary series $\lambda_j(w)$, $j = 1, 2, 3$. Up to a rescaling $T(w) \rightarrow \lambda_2^{-1}(w)T(w)$, we will only need the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \quad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}.$$

where r_1 and r_3 are free functional parameters.

1.4 Aim

The purpose in integrable systems is twofold:

1. Compute the Bethe vectors (BVs), eigenvectors of $t(x)$

$$t(x) \mathbb{B}^{a,b}(\bar{u}, \bar{v}) = \tau(x|\bar{u}, \bar{v}) \mathbb{B}^{a,b}(\bar{u}, \bar{v}).$$

This part is well-understood and is done using the algebraic Bethe ansatz method. It leads to the celebrated Bethe ansatz eqs (BAE).

2. Compute correlation functions $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ for some local operators \mathcal{O}_j . This calculation can be decomposed in four steps:
 - (a) Express the operators \mathcal{O}_j in terms of monodromy entries $T_{kl}(x)$;
 - (b) Action of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u}, \bar{v})$;
 - (c) Scalar product of off-shell BVs (without BAE);
 - (d) Form factors $\mathbb{C}^{a,b}(\bar{t}, \bar{s}) T_{ij}(\bar{x}) \mathbb{B}^{a,b}(\bar{u}, \bar{v})$.

In part 2, one needs to find *simple* (i.e. factorized) expressions in order to be able to take the thermodynamical limit and extract the asymptotic behavior of the correlation functions.

Here, we will present these two parts for the model based on $Y(\mathfrak{gl}_3)$. The calculations are rather technical, so that we will present here the results only, and refer to the original papers for the complete calculations. The presentation follows the plan explained above, and we will show how the techniques apply for other models in the conclusion.

2 Notation

Apart from the functions $g(x, y) = \frac{c}{x-y}$, $r_1(x)$ and $r_3(x)$ introduced above, we note

$$f(x, y) = \frac{x - y + c}{x - y}, \quad h(x, y) = \frac{f(x, y)}{g(x, y)}, \quad t(x, y) = \frac{g(x, y)}{h(x, y)}.$$

Clearly $f(x, y) = 1 + g(x, y)$ but this identification is not true for the q -analogues of these functions, so we keep this distinction.

To make presentation lighter, we will use the following conventions:

- "bar" always denote sets of variables: \bar{w} , \bar{u} , \bar{v} etc.
- $|\cdot|$ is the dimension of a set: $\bar{w} = \{w_1, w_2\} \Rightarrow |\bar{w}| = 2$, etc.
- Individual elements of the sets have latin subscripts: w_j , u_k , etc.
- Subsets of variables are denoted by roman indices: \bar{u}_I , \bar{v}_{IV} , \bar{w}_{II} , etc.
- Special case: $\bar{u}_j = \bar{u} \setminus \{u_j\}$, $\bar{w}_k = \bar{w} \setminus \{w_k\}$, etc.

We will also use shorthand notations for products of scalar functions:

$$f(\bar{u}_{II}, \bar{u}_I) = \prod_{u_j \in \bar{u}_{II}} \prod_{u_k \in \bar{u}_I} f(u_j, u_k), \quad r_1(\bar{u}_{II}) = \prod_{u_j \in \bar{u}_{II}} r_1(u_j), \quad g(v_k, \bar{w}) = \prod_{w_j \in \bar{w}} g(v_k, w_j), \quad \text{etc.}$$

3 Bethe vectors

The framework for the construction of Bethe vectors is the Nested Bethe ansatz as introduced in [4]. This technics is well-known, but the explicit expressions for these BVs are rather recent, so we briefly remind them here.

3.1 On-shell Bethe vectors

The Bethe vectors $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$ depend on two sets of parameters $\bar{u} = \{u_1, \dots, u_a\}$ and $\bar{v} = \{v_1, \dots, v_b\}$. The superscripts a and b in \mathbb{B} indicate the cardinalities of the sets, $|\bar{u}| = a$ and $|\bar{v}| = b$. They are eigenvectors of the transfer matrix

$$t(x) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \tau(x|\bar{u}; \bar{v}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}), \quad (2)$$

$$\tau(x|\bar{u}; \bar{v}) = r_1(w)f(\bar{u}, w) + f(w, \bar{u})f(\bar{v}, w) + r_3(w)f(w, \bar{v}), \quad (3)$$

provided \bar{u} and \bar{v} obey the Bethe equations (BAEs):

$$r_1(\bar{u}_I) = \frac{f(\bar{u}_I, \bar{u}_{II})}{f(\bar{u}_{II}, \bar{u}_I)} f(\bar{v}, \bar{u}_I), \quad (4)$$

$$r_3(\bar{v}_I) = \frac{f(\bar{v}_I, \bar{v}_{II})}{f(\bar{v}_{II}, \bar{v}_I)} f(\bar{v}_I, \bar{u}). \quad (5)$$

that hold for arbitrary partitions of the sets \bar{u} and \bar{v} into subsets $\{\bar{u}_I, \bar{u}_{II}\}$ and $\{\bar{v}_I, \bar{v}_{II}\}$. In that case, the BVs will be called "on-shell", while they will be called "off-shell" if the BAEs are not obeyed. Of course, in that latter case, the BVs are not eigenvectors of $t(x)$.

3.2 Dual Bethe vectors $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$, $|\bar{u}| = a$, $|\bar{v}| = b$

Dual BVs are constructed as left eigenvectors of the transfer matrix:

$$\mathbb{C}^{a,b}(\bar{u}; \bar{v}) t(x) = \tau(x|\bar{u}; \bar{v}) \mathbb{C}^{a,b}(\bar{u}; \bar{v}), \quad (6)$$

where the Bethe parameters \bar{u}, \bar{v} obey the BAEs (4)–(5). Again, these dual BVs will be called on-shell when \bar{u} and \bar{v} obey the BAEs, while they will be called off-shell dual BVs when \bar{u}, \bar{v} are left free.

3.3 Trace formula

This is a known and quite general formula, given in [5] for \mathfrak{gl}_N and $\mathcal{U}_q(\mathfrak{gl}_N)$ algebras, and generalized in [6] for superalgebras. It expresses $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$ as a trace in $a + b$ auxiliary spaces of products of monodromy matrices:

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \text{tr} \left(\mathbb{T}(\bar{u}; \bar{v}) \mathbb{R}(\bar{u}; \bar{v}) e_{21}^{\otimes a} \otimes e_{32}^{\otimes b} \right) \in Y(\mathfrak{gl}_3), \quad (7)$$

where \mathbb{T} is some product of monodromy matrices $T(x)$ and \mathbb{R} some product of R -matrices. Their explicit expression can be found in [5, 6].

3.4 Recursion formulas

It can be shown that the Bethe vectors also obey the following recursion relations [7]:

$$\lambda_2(u_k)f(\bar{v}, u_k)\mathbb{B}^{a+1,b}(\bar{u}; \bar{v}) = T_{12}(u_k)\mathbb{B}^{a,b}(\bar{u}_k; \bar{v}) + \sum_{i=1}^b g(v_i, u_k)f(\bar{v}_i, v_i)T_{13}(u_k)\mathbb{B}^{a,b-1}(\bar{u}_k; \bar{v}_i), \quad (8)$$

$$\lambda_2(v_k)f(v_k, \bar{u})\mathbb{B}^{a,b+1}(\bar{u}; \bar{v}) = T_{23}(v_k)\mathbb{B}^{a,b}(\bar{u}; \bar{v}_k) + \sum_{j=1}^a g(v_k, u_j)f(u_j, \bar{u}_j)T_{13}(v_k)\mathbb{B}^{a-1,b}(\bar{u}_j; \bar{v}_k). \quad (9)$$

Let us remark that (8) completely determines the Bethe vectors once $\mathbb{B}^{0,b}(\emptyset; \bar{v})$ is known. In the same way, (9) completely determines the Bethe vectors once $\mathbb{B}^{a,0}(\bar{u}; \emptyset)$ is fixed.

3.5 Explicit formulas

There is a third series of expressions for Bethe vectors, using partitions of \bar{u} and \bar{v} [7]:

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{\mathsf{K}_k(\bar{v}_I|\bar{u}_I)}{\lambda_2(\bar{v}_I)\lambda_2(\bar{u})} \frac{f(\bar{v}_I, \bar{v}_I)f(\bar{u}_I, \bar{u}_I)}{f(\bar{v}_I, \bar{u})f(\bar{v}_I, \bar{u}_I)} T_{12}(\bar{u}_I)T_{13}(\bar{u}_I)T_{23}(\bar{v}_I) |0\rangle, \quad (10)$$

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{\mathsf{K}_k(\bar{v}_I|\bar{u}_I)}{\lambda_2(\bar{u}_I)\lambda_2(\bar{v})} \frac{f(\bar{v}_I, \bar{v}_I)f(\bar{u}_I, \bar{u}_I)}{f(\bar{v}_I, \bar{u}_I)f(\bar{v}, \bar{u}_I)} T_{23}(\bar{v}_I)T_{13}(\bar{v}_I)T_{12}(\bar{u}_I) |0\rangle, \quad (11)$$

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{\mathsf{K}_k(\bar{v}_I|\bar{u}_I)}{\lambda_2(\bar{v}_I)\lambda_2(\bar{u})} \frac{f(\bar{v}_I, \bar{v}_I)f(\bar{u}_I, \bar{u}_I)}{f(\bar{v}, \bar{u})} T_{13}(\bar{u}_I)T_{12}(\bar{u}_I)T_{23}(\bar{v}_I) |0\rangle, \quad (12)$$

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{\mathsf{K}_k(\bar{v}_I|\bar{u}_I)}{\lambda_2(\bar{u}_I)\lambda_2(\bar{v})} \frac{f(\bar{v}_I, \bar{v}_I)f(\bar{u}_I, \bar{u}_I)}{f(\bar{v}, \bar{u})} T_{13}(\bar{v}_I)T_{23}(\bar{v}_I)T_{12}(\bar{u}_I) |0\rangle. \quad (13)$$

The sums are taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_I\}$ and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_I\}$ with the condition $0 \leq |\bar{u}_I| = |\bar{v}_I| = k \leq \min(a, b)$.

$\mathsf{K}_k(\bar{v}_I|\bar{u}_I)$ is the Izergin–Korepin determinant [8]

$$\mathsf{K}_k(\bar{x}|\bar{y}) = \prod_{\ell < m}^k g(x_\ell, x_m)g(y_m, y_\ell) \cdot h(\bar{x}, \bar{y}) \det_k [t(x_i, y_j)]. \quad (14)$$

3.6 All these formulas are related

Let us stress that all the above formulas define the same Bethe vectors, should they be on-shell or off-shell. For instance, one can show that

- The explicit expressions obey the recursion formulas;
- The trace formula obeys the recursion formulas too;
- Recursion formulas can be obtained starting from the trace formula.

Depending on the calculation, one can then freely choose any of these expression to prove a formula or a property of BVs.

4 Correlation functions

We now turn to the second step of our program, that is, for a local operator \mathcal{O} , how to compute its mean value? As a first step, we are led with the following question:

How to compute $\mathcal{O}_{\mathbb{C},\mathbb{B}} = \langle \mathbb{C} | \mathcal{O} | \mathbb{B} \rangle$?

Assuming that $\{|\mathbb{B}\rangle\}$ forms a complete basis (of transfer matrix eigenspaces), we have

$$\mathcal{O}|\mathbb{B}\rangle = \sum_{\mathbb{B}'} \mathbb{O}_{\mathbb{B}\mathbb{B}'} |\mathbb{B}'\rangle, \quad (15)$$

so that we "only" need $\langle \mathbb{C} | \mathbb{B}' \rangle$ and of course the decomposition (15).

Now, for a spin chain of length L and based on \mathfrak{gl}_N -fundamental representations, local operators have a decomposition²

$$\mathcal{O} = \sum_{\ell=1}^L \sum_{i,j=1}^N \mathcal{O}_{ij}^{(\ell)} e_{ij}^{\ell}, \quad (16)$$

where e_{ij}^{ℓ} is the elementary matrix e_{ij} at site ℓ . Then, everything boils down to the calculation of $\langle \mathbb{C} | e_{ij}^{\ell} | \mathbb{B} \rangle$.

A further simplification occurs because of QISM. Indeed, the expression of e_{ij}^{ℓ} , $i, j = 1, 2, \dots, N$ and $\ell = 1, \dots, L$ is known in terms of monodromy entries $T_{kl}(x)$, $k, l = 1, \dots, N$ [9]:

$$e_{ij}^{\ell} = (t(0))^{\ell-1} T_{ij}(0) (t(0))^{-\ell}. \quad (17)$$

Then, from (16) and (17), if we can compute $T_{kl}(x) \mathbb{B}^{a,b}(\bar{u}; \bar{v})$ and $\mathbb{C}^{a,b}(\bar{w}; \bar{z}) \mathbb{B}^{a,b}(\bar{u}; \bar{v})$, we are able to compute any correlation function. The following sections are devoted to the calculation of these two fundamental quantities in the case $N = 3$.

5 Multiple actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$

Using the explicit expressions of section 3.5, we were able in [10] to compute explicitly the actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$. Denoting $\{\bar{u}, \bar{x}\} = \bar{\eta}$, $\{\bar{v}, \bar{x}\} = \bar{\xi}$ and the cardinalities by $|\bar{x}| = n$, $|\bar{\eta}| = a + n$ and $|\bar{\xi}| = b + n$, we have

$$T_{13}(\bar{x}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{x}) \mathbb{B}^{a+n, b+n}(\bar{\eta}; \bar{\xi}), \quad (18)$$

$$T_{12}(\bar{x}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{x}) \sum f(\bar{\xi}_{\text{II}}, \bar{\xi}_{\text{I}}) \mathbf{K}_n(\bar{\xi}_{\text{I}} | \bar{x} + c) \mathbb{B}^{a+n, b}(\bar{\eta}; \bar{\xi}_{\text{II}}), \quad (19)$$

$$T_{23}(\bar{x}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{x}) \sum f(\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}}) \mathbf{K}_n(\bar{x} | \bar{\eta}_{\text{I}} + c) \mathbb{B}^{a, b+n}(\bar{\eta}_{\text{II}}; \bar{\xi}). \quad (20)$$

In (19), the sum is on partitions $\bar{\xi} = \{\bar{\xi}_{\text{I}}, \bar{\xi}_{\text{II}}\}$ with $|\bar{\xi}_{\text{I}}| = n$, while in (20), the sum is on partitions $\bar{\eta} = \{\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}}\}$ with $|\bar{\eta}_{\text{I}}| = n$. Similar expressions can be obtained for any $T_{ij}(\bar{x})$ and for dual BVs, see [10].

²The same ideas can be applied for a general spin chain, using an adapted basis.

Remark that the relations (19) and (20) imply recursion relations of section 3.4 as a subcase (for $n = 1$).

Since the action of $T_{ij}(\bar{x})$ operators on BVs gives back BVs (that are a priori off-shell), it remains to compute scalar products of BVs to get the full form factor expression.

6 Scalar products of BVs

In this section, we provide expression for the scalar product

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{u}^B | \bar{v}^C, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (21)$$

where $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are general (dual) BVs. Let us stress that the superscripts B and C are used to denote *different* sets of (Bethe) parameters, completely independent one from each other.

6.1 Reshetikhin's formula

There is a well-known formula, due to Reshetikhin [11], and valid for \mathfrak{gl}_N :

$$\begin{aligned} \mathcal{S}_{a,b} = & \sum r_1(\bar{u}_I^B) r_1(\bar{u}_{II}^C) r_3(\bar{v}_I^B) r_3(\bar{v}_{II}^C) f(\bar{u}_I^C, \bar{u}_{II}^C) f(\bar{u}_{II}^B, \bar{u}_I^B) f(\bar{v}_{II}^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_{II}^B) f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_{II}^B, \bar{u}_{II}^B) \\ & \times Z_{a-k,n}(\bar{u}_{II}^C; \bar{u}_{II}^B | \bar{v}_I^C; \bar{v}_I^B) Z_{k,b-n}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_{II}^B; \bar{v}_{II}^C), \end{aligned} \quad (22)$$

where the sum is on partitions $\bar{u}^B = \{\bar{u}_I^B, \bar{u}_{II}^B\}$, $\bar{u}^C = \{\bar{u}_I^C, \bar{u}_{II}^C\}$ with $|\bar{u}_I^B| = |\bar{u}_I^C| = k$ for $k = 0, \dots, a$, $\bar{v}^B = \{\bar{v}_I^B, \bar{v}_{II}^B\}$, $\bar{v}^C = \{\bar{v}_I^C, \bar{v}_{II}^C\}$ with $|\bar{v}_I^B| = |\bar{v}_I^C| = n$ for $n = 0, \dots, b$.

$Z_{a,b}$ are the so-called highest coefficients

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = (-1)^b \sum K_b(\bar{s} - c | \bar{w}_I) K_a(\bar{w}_{II} | \bar{t}) K_b(\bar{y} | \bar{w}_I) f(\bar{w}_I, \bar{w}_{II}), \quad (23)$$

where the sum is done over partitions of $\bar{w} = \{\bar{s}, \bar{x}\}$ into subsets \bar{w}_I and \bar{w}_{II} with $|\bar{w}_I| = b$.

The formula is valid for a general scalar product, but as it stands, $\mathcal{S}_{a,b}$ is difficult to handle. To compute e.g. the thermodynamical limit of such formula, and to use it for the calculation of correlation functions, one needs to find a factorized form, containing only one determinant. It was done for the \mathfrak{gl}_2 case [12], but for \mathfrak{gl}_3 (and a fortiori for \mathfrak{gl}_N) no such formula is known yet. However, in some particular cases, there exists such a formula:

1. When computing the norm of a Bethe vector that is assumed to be on-shell, such an expression was obtained by Reshetikhin [11];
2. A nice factorized expression was obtained in [13], when some of the Bethe parameters tend to infinity;
3. When the BV is on-shell and the dual BV is "twisted on-shell" (see below), we were able to get a simplified expression [7];
4. In [14], we provided different expressions for the highest coefficients (23);
5. An interesting multiple integral expression for the scalar product of an on-shell and an off-shell BV was recently obtained in [15].

We present the points 3 and 4 in the two following sections.

6.2 Highest coefficients

Highest coefficients were introduced by Reshetekhin [11] and play a central role in the expression of the scalar product of Bethe vectors. In fact, they can be viewed as partition functions of a statistical models with some particular boundary conditions. It is thus important to get different forms for them. We give here some examples of such formulas, a more complete list can be found in [14].

Sums on partitions. There are different series of expressions for the highest coefficients. A first series is given by sums over partitions. The expression (23) is a first example of such formulas. Another example is given by

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = (-1)^a f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t}) \sum K_a(\bar{t} - c | \bar{\eta}_I) K_a(\bar{x} | \bar{\eta}_I) K_b(\bar{\eta}_{II} - c | \bar{s}) f(\bar{\eta}_I, \bar{\eta}_{II}), \quad (24)$$

where $\bar{\eta} = \{\bar{y} + c, \bar{t}\}$. The sum is taken with respect to partitions of the set $\bar{\eta}$ into subsets $\bar{\eta}_I$ and $\bar{\eta}_{II}$ with $\#\bar{\eta}_I = a$.

Recursion formulas. The most important property of the highest coefficient $Z_{a,b}$ is that its residues in its poles can be expressed in terms of $Z_{a-1,b}$ or $Z_{a,b-1}$. Since $Z_{a,b}$ is a rational function in all its variables, this property allows us to fix it unambiguously, provided we know some initial condition. It is easy to see that for $a = 0$ or $b = 0$ $Z_{a,b}$ coincides with K_n :

$$Z_{a,0}(\bar{t}; \bar{x} | \emptyset; \emptyset) = K_a(\bar{x} | \bar{t}), \quad Z_{0,b}(\emptyset; \emptyset | \bar{s}; \bar{y}) = K_b(\bar{y} | \bar{s}). \quad (25)$$

Consider $Z_{a,b}$ as a function of s_b with all other variables fixed. Then it has simple poles at $s_b = y_m$, $m = 1, \dots, b$ and $s_b = t_\ell$, $\ell = 1, \dots, a$. Due to the symmetry of $Z_{a,b}$ over \bar{y} and over \bar{t} it is enough to find the residues at $s_b = y_b$ and $s_b = t_a$. These residues are given by:

$$\text{Res } Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) \Big|_{s_b=y_b} = -c f(y_b, \bar{s}_b) f(\bar{y}_b, y_b) f(y_b, \bar{x}) Z_{a,b-1}(\bar{t}; \bar{x} | \bar{s}_b; \bar{y}_b), \quad (26)$$

$$\text{Res } Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) \Big|_{s_b=t_a} = c f(\bar{s}_b, t_a) f(t_a, \bar{t}_a) \sum_{p=1}^a g(x_p, t_a) f(\bar{x}_p, x_p) Z_{a-1,b}(\bar{t}_a; \bar{x}_p | \{\bar{s}_b, x_p\}; \bar{y}_b), \quad (27)$$

where $\bar{s}_b = \bar{s} \setminus s_b$, $\bar{y}_b = \bar{y} \setminus y_b$, etc.

Contour integral. There exists several representations for $Z_{a,b}$ in terms of multiple contour integrals of Cauchy type. Here, we give only one possible integral as example:

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = \frac{1}{(2\pi i c)^b b!} \oint_{\bar{w}} K_b(\bar{s} - c | \bar{z}) K_b(\bar{y} | \bar{z}) K_{a+b}(\bar{w} | \bar{t}, \bar{z} + c) f(\bar{z}, \bar{w}) \mathcal{F}_b(\bar{z}) d\bar{z}, \quad (28)$$

where we have a b -fold integral and

$$\mathcal{F}_b(\bar{z}) = \prod_{j=1}^b f^{-1}(z_j, \bar{z}_j).$$

Other expressions of the type (28), or implying a -fold integrals can be found in [14].

6.3 Scalar product for twisted Bethe vectors

Here we consider an on-shell Bethe vector, eigenvector of the transfer matrix

$$t(x) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \tau(x|\bar{u}^B, \bar{v}^B) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (29)$$

where the Bethe parameters $\{\bar{u}^B; \bar{v}^B\}$ obey the Bethe equations (4)-(5). We also introduce, for any complex number κ , a twisted transfer matrix

$$t_\kappa(x) = T_{11}(x) + \kappa T_{22}(x) + T_{33}(x) = \text{tr}(M T(x)) \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (30)$$

and its *twisted* dual on-shell Bethe vector

$$\mathbb{C}_\kappa^{a,b}(\bar{u}^C; \bar{v}^C) t_\kappa(x) = \tau_\kappa(x|\bar{u}^C, \bar{v}^C) \mathbb{C}_\kappa^{a,b}(\bar{u}^C; \bar{v}^C). \quad (31)$$

It is an eigenvector of $t_\kappa(x)$ when the Bethe parameters \bar{u}^C, \bar{v}^C obey the *twisted* BAEs

$$r_1(\bar{u}_I) = \kappa \frac{f(\bar{u}_I, \bar{u}_{II})}{f(\bar{u}_{II}, \bar{u}_I)} f(\bar{v}, \bar{u}_I), \quad (32)$$

$$r_3(\bar{v}_I) = \kappa \frac{f(\bar{v}_I, \bar{v}_{II})}{f(\bar{v}_{II}, \bar{v}_I)} f(\bar{v}_I, \bar{u}). \quad (33)$$

Let us stress that the superscripts B and C are there to distinguish the Bethe parameters of \mathbb{B}^{ab} from those of \mathbb{C}^{ab} . In other words, the Bethe parameters $\{\bar{u}^B, \bar{v}^B\}$ are a priori not related to $\{\bar{u}^C, \bar{v}^C\}$.

In [7], we obtained an expression for the scalar product

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{u}^B | \bar{v}^C, \bar{v}^B) = \mathbb{C}_\kappa^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B). \quad (34)$$

Indeed, the scalar product can be written as

$$\mathcal{S}_{a,b} = f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B) t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det_{a+b} \mathcal{N}, \quad (35)$$

where

$$\Delta'_n(\bar{x}) = \prod_{j>k}^n g(x_j, x_k), \quad \Delta_n(\bar{y}) = \prod_{j<k}^n g(y_j, y_k).$$

and \mathcal{N} is a block-matrix of the size $(a+b) \times (a+b)$,

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{(u)}(u_j^C, u_k^B) & \mathcal{N}^{(u)}(u_j^C, v_k^C) \\ \mathcal{N}^{(v)}(v_j^B, u_k^B) & \mathcal{N}^{(v)}(v_j^B, v_k^C) \end{pmatrix} = \left(\begin{array}{c|c} a \times a & a \times b \\ \hline b \times a & b \times b \end{array} \right),$$

whose full expression is given in appendix A. We show below how this expression can give rise to a factorized expression for form factors of the model.

Expression for a general twist

A similar expression for $\mathcal{S}_{a,b}$ can be obtained when considering a general twist $\bar{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$ of the transfer matrix

$$t_{\bar{\kappa}}(x) = \kappa_1 T_{11}(x) + \kappa_2 T_{22}(x) + \kappa_3 T_{33}(x).$$

However, in that case, the expression is valid only up to terms $(\kappa_i - 1)(\kappa_j - 1)$, $i, j = 1, 2, 3$, that are irrelevant for our purpose, as we shall see below. For further application it is useful to write the system of twisted Bethe equations in the logarithmic form. Let us define

$$\Phi_j = \log r_1(u_j^C) - \log \left(\frac{f(u_j^C, \bar{u}_j^C)}{f(\bar{u}_j^C, u_j^C)} \right) - \log f(\bar{v}^C, u_j^C), \quad j = 1, \dots, a, \quad (36)$$

and

$$\Phi_{j+a} = \log r_3(v_j^C) - \log \left(\frac{f(\bar{v}_j^C, v_j^C)}{f(v_j^C, \bar{v}_j^C)} \right) - \log f(v_j^C, \bar{u}^C), \quad j = 1, \dots, b. \quad (37)$$

Then the system of twisted Bethe equations for general $\bar{\kappa}$ takes the form

$$\begin{aligned} \Phi_j &= \log \kappa_2 - \log \kappa_1 + 2\pi i \ell_j, & j &= 1, \dots, a, \\ \Phi_{j+a} &= \log \kappa_2 - \log \kappa_3 + 2\pi i m_j, & j &= 1, \dots, b, \end{aligned} \quad (38)$$

where ℓ_j and m_j are some integers.

7 Form factors

We present now the calculation [16] the form factor of the diagonal elements $T_{ss}(z)$

$$\mathcal{F}_{a,b}^{(s)}(z) \equiv \mathcal{F}_{a,b}^{(s)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ss}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (39)$$

where both $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors. Form factors for off-diagonal elements $T_{j,j+1}(z)$ and $T_{j+1,j}(z)$ have been given in [17]. The form factors associated to $T_{13}(z)$ and $T_{31}(z)$ remain to be done. Of course, the ultimate goal would be to find a simple expression for the form factor when the Bethe vector $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ and/or the dual Bethe vector $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$ are off-shell. Up to now, such an expression is still missing.

A priori, from the knowledge of the actions (18)-(20) and the scalar products (22), we can deduce an expression of the form factor. However, the expression is rather complicated and difficult to handle. Fortunately, one can get another simpler form using the following trick.

Let us consider $\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)$ a twisted on-shell Bethe vector such that

$$\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)|_{\bar{\kappa}=1} = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C). \quad (40)$$

Then, the form factor (39) can be expressed as

$$\mathcal{F}^{(s)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \frac{dQ_{\bar{\kappa}}(z)}{d\kappa_s} \Big|_{\bar{\kappa}=1}, \quad s = 1, 2, 3$$

where $\bar{\kappa} = 1$ means $\kappa_1 = \kappa_2 = \kappa_3 = 1$ and

$$\begin{aligned} Q_{\bar{\kappa}}(z) &= \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)(t_{\bar{\kappa}}(z) - t(z))\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \\ &= (\tau_{\kappa}(z|\bar{u}^C, \bar{v}^C) - \tau(z|\bar{u}^B, \bar{v}^B))\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B). \end{aligned}$$

Then, it is clear that all depends on the expression of the scalar product $\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$, and that we need to know this scalar product only up to terms $(\kappa_i - 1)(\kappa_j - 1)$, $i, j = 1, 2, 3$. Depending on whether $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$ is $(\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^{\dagger}$ or not, we get two different expressions:

When $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^{\dagger}$

$$\begin{aligned} \mathcal{F}^{(s)}(z|\bar{u}, \bar{v}; \bar{u}, \bar{v}) &= \|\mathbb{B}^{a,b}(\bar{u}; \bar{v})\|^2 \frac{d\tau_{\bar{\kappa}}(z|\bar{u}^C; \bar{v}^C)}{d\kappa_s} \Big|_{\bar{\kappa}=1} \\ &= (-1)^a c^{a+b} f(\bar{v}, \bar{u}) \prod_{j=1}^a f(u_j, \bar{u}_j) \prod_{k=1}^b f(v_k, \bar{v}_k) \det_{a+b+1} \Theta^{(s)}(z), \end{aligned} \quad (41)$$

where $\Theta^{(s)}(z)$ is an $(a+b+1) \times (a+b+1)$ matrix given in appendix B.

When $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^{\dagger}$

$$\begin{aligned} \mathcal{F}_{a,b}^{(s)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) &= \left((\tau(z|\bar{u}^C; \bar{v}^C) - \tau(z|\bar{u}^B; \bar{v}^B)) \frac{d}{d\kappa_s} \left(\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \right) \right) \Big|_{\bar{\kappa}=1} \\ &= \frac{\tau(z|\bar{u}^C; \bar{v}^C) - \tau(z|\bar{u}^B; \bar{v}^B)}{\Omega_p} t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \\ &\quad \times \det_{a+b} \mathcal{N}^{(s,p)}, \end{aligned}$$

The integer p is such that $\Omega_p \neq 0$ where

$$\begin{aligned} \Omega_k &= \prod_{\ell=1}^a (u_k^C - u_{\ell}^B) \prod_{\substack{\ell=1 \\ \ell \neq k}}^a (u_k^C - u_{\ell}^C)^{-1}, & k = 1, \dots, a, \\ \Omega_{a+k} &= \prod_{m=1}^b (v_k^B - v_m^C) \prod_{\substack{m=1 \\ m \neq k}}^b (v_k^B - v_m^B)^{-1}, & k = 1, \dots, b. \end{aligned} \quad (42)$$

The matrix $\mathcal{N}^{(s,p)}$ has a special p^{th} row, but its determinant is independent of p . The form of $\mathcal{N}^{(s,p)}$ is given in appendix C.

8 Conclusion

For models with a \mathfrak{gl}_3 invariant R -matrix, we have presented several explicit expressions for (off-shell) Bethe vectors and duals BVs. We have also computed the multiple action of monodromy elements on

these BVs. Both results are presented in term of Izergin-Korepin determinants and sums over partitions of sets of Bethe parameters.

In a second step, we calculated the scalar product of (twisted) on-shell BVs and the form factors of $T_{ss}(x)$, $s = 1, 2, 3$, of $T_{j,j+1}(x)$ and of $T_{j+1,j}(x)$, $j = 1, 2$. Both results were given in term of a single determinant (and product of scalar functions).

The ultimate goal is to obtain a single determinant expression for the correlation functions of the model, so as to study the thermodynamical limit and their asymptotics. Of course, to get to that point a lot remains to be done. For instance, it remains to compute the form factors of $T_{13}(x)$ and $T_{31}(x)$. The calculation of the scalar product of generic off-shell BVs (as a single determinant) is also lacking.

Certainly, a generalization to other integrable models is wanted. As a first step, we started to investigate the case of \mathfrak{gl}_3 XXZ spin chain (i.e. based on the R -matrix of $\mathcal{U}_q(\mathfrak{gl}_3)$):

1. The multiple action of $T_{ij}(x)$ generators on BVs was performed in [18];
2. The calculation of the highest coefficient was done in [19];
3. A Reshetikhin-like formula for scalar products of the $\mathcal{U}_q(\mathfrak{gl}_3)$ model is given in [20].

Let us remark that to obtain these results, we used the current realization of $\mathcal{U}_q(\mathfrak{gl}_3)$ and the construction of Khoroshkin, Pakuliak and collaborators for BVs in this presentation [21]. This construction is valid for $\mathcal{U}_q(\mathfrak{gl}_N)$: a link between the current presentation of BVs and the explicit expression of BVs using the monodromy matrix for $\mathcal{U}_q(\mathfrak{gl}_N)$ is done in [22]. The use of a morphism between $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathcal{U}_{q^{-1}}(\mathfrak{gl}_N)$ is essential in this construction.

A The matrix \mathcal{N}

Diagonal blocks

$$\begin{aligned} \mathcal{N}^{(u)}(u_j^C, u_k^B) &= h(\bar{v}^C, u_k^B) h(u_k^B, \bar{u}^C) \left[\kappa t(u_k^B, u_j^C) \right. \\ &\quad \left. + t(u_j^C, u_k^B) \frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)} \frac{h(\bar{u}^C, u_k^B) h(u_k^B, \bar{u}^B)}{h(u_k^B, \bar{u}^C) h(\bar{u}^B, u_k^B)} \right] \quad a \times a \text{ block} \end{aligned}$$

$$\begin{aligned} \mathcal{N}^{(v)}(v_j^B, v_k^C) &= h(v_k^C, \bar{u}^B) h(\bar{v}^B, v_k^C) \left[t(v_j^B, v_k^C) \right. \\ &\quad \left. + \kappa t(v_k^C, v_j^B) \frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)} \frac{h(\bar{v}^C, v_k^C) h(v_k^C, \bar{v}^B)}{h(v_k^C, \bar{v}^C) h(\bar{v}^B, v_k^C)} \right] \quad b \times b \text{ block} \end{aligned}$$

Off-diagonal blocks

$$\begin{aligned} \mathcal{N}^{(u)}(u_j^C, v_k^C) &= \kappa t(v_k^C, u_j^C) h(v_k^C, \bar{u}^C) h(\bar{v}^C, v_k^C) \quad a \times b \text{ block} \\ \mathcal{N}^{(v)}(v_j^B, u_k^B) &= t(v_j^B, u_k^B) h(\bar{v}^B, u_k^B) h(u_k^B, \bar{u}^B) \quad b \times a \text{ block} \end{aligned}$$

B The matrix $\Theta^{(s)}$

First of all we define an $(a+b) \times (a+b)$ matrix θ with the entries

$$\theta_{j,k} = \frac{\partial \Phi_j}{\partial u_k^C} \Big|_{\substack{\bar{u}^C = \bar{u} \\ \bar{v}^C = \bar{v}}}, \quad k = 1, \dots, a; \quad \text{and} \quad \theta_{j,k+a} = \frac{\partial \Phi_j}{\partial v_k^C} \Big|_{\substack{\bar{u}^C = \bar{u} \\ \bar{v}^C = \bar{v}}}, \quad k = 1, \dots, b, \quad (43)$$

where the Φ_j are given by (36) and (37).

Then we extend the matrix θ to an $(a+b+1) \times (a+b+1)$ matrix $\Theta^{(s)}$ with $s = 1, 2, 3$, by adding one row and one column

$$\begin{aligned} \Theta_{j,k}^{(s)} &= \theta_{j,k}, & j, k &= 1, \dots, a+b, \\ \Theta_{a+b+1,k}^{(s)} &= \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial u_k^C}, & k &= 1, \dots, a, & \Theta_{a+b+1,a+k}^{(s)} &= \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial v_k^C}, & k &= 1, \dots, b, \\ \Theta_{j,a+b+1}^{(s)} &= \delta_{s1} - \delta_{s2} & j &= 1, \dots, a, & \Theta_{j+a,a+b+1}^{(s)} &= \delta_{s3} - \delta_{s2} & j &= 1, \dots, b, \\ \Theta_{a+b+1,a+b+1}^{(s)} &= \frac{\partial \tau_{\bar{\kappa}}(z|\bar{u}^C, \bar{v}^C)}{\partial \kappa_s} \Big|_{\substack{\bar{u}^C = \bar{u} \\ \bar{v}^C = \bar{v}}}. \end{aligned}$$

Here the δ_{sk} are Kronecker deltas. Notice that $\Theta^{(s)}$ depends on s only in its last column.

C The matrix $\mathcal{N}^{(s,p)}$

For $j \neq p$ we define the entries $\mathcal{N}_{j,k}^{(s,p)}$ of the $(a+b) \times (a+b)$ matrix $\mathcal{N}^{(s,p)}$ as

$$\mathcal{N}_{j,k}^{(s)} = c g^{-1}(w_k, \bar{u}^C) g^{-1}(\bar{v}^C, w_k) \frac{\partial \tau(w_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C}, \quad j = 1, \dots, a, \quad j \neq p, \quad (44)$$

$$\mathcal{N}_{a+j,k}^{(s)} = -c g^{-1}(\bar{v}^B, w_k) g^{-1}(w_k, \bar{u}^B) \frac{\partial \tau(w_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B}, \quad j = 1, \dots, b, \quad j \neq p. \quad (45)$$

In these formulas one should set $w_k = u_k^B$ for $k = 1, \dots, a$ and $w_{k+a} = v_k^C$ for $k = 1, \dots, b$.

The p -th row has the following elements

$$\mathcal{N}_{p,k}^{(s)} = h(\bar{v}^C, w_k) h(w_k, \bar{u}^B) Y_k^{(s)}, \quad (46)$$

where again $w_k = u_k^B$ for $k = 1, \dots, a$ and $w_{k+a} = v_k^C$ for $k = 1, \dots, b$, and

$$\begin{aligned} Y_k^{(s)} &= c(\delta_{s1} - \delta_{s2}) + (\delta_{s1} - \delta_{s3}) u_k^B \left(1 - \frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)} \right), & k &= 1, \dots, a, \\ Y_{a+k}^{(s)} &= c(\delta_{s3} - \delta_{s2}) + (\delta_{s1} - \delta_{s3}) (v_k^C + c) \left(1 - \frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)} \right), & k &= 1, \dots, b. \end{aligned} \quad (47)$$

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